**Indian Statistical Institute, Bangalore**
**B. Math II, First Semester, 2022-23**
**Final Examination, Introduction to Statistical Inference**

**21.11.22** **Maximum Score 100** **Duration: 3 Hours**

1. (5+5+5+5+5) We have data $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ where all the random variables are independent, $X_i$ follows Bernoulli($p$) and $Y_j$ follows Geometric($p$). Let $I_X(p)$ be the information on $p$ based on $X_1$ and $I_Y(p)$ be the information based on $Y_1$.
   (a) Show that $T=(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j)$ is sufficient for $p$.
   (b) Show that the distribution of $T$ belongs to a curved exponential family.
   (c) Find the MLE of $p$.
   (d) Find $I(p)$, the information on $p$ based on the available data.
   (e) Find $I_X(p)$ and $I_Y(p)$. Express $I(p)$ as a linear combination of $I_X(p)$ and $I_Y(p)$.

2. (5+5+10) Consider the following regression model.
$$y_i = bx_i + e_i, \quad 1 \le i \le n,$$
   where $x_i$'s are fixed non-zero real numbers and $e_i$'s are independent random variables with mean zero and equal variance.
   (a) Find the least squares estimator of $b$, that is, the value of $b$ that minimizes $\sum_{i=1}^n (y_i - bx_i)^2$.
   (b) Consider a general linear estimator of the form $\sum_{i=1}^n a_i y_i$. Find the mean and variance of this estimator.
   (c) Show that among all unbiased linear estimators, the least squares estimator has the lowest variance.

3. (10+10) Let $X_1, \cdots, X_n$ be iid $\mathcal{N}(\mu_1, \sigma^2)$ random variables and $Y_1, \cdots, Y_m$ be and independent set of iid $\mathcal{N}(\mu_2, \sigma^2)$ random variables.
   (a) Derive the generalized likelihood ratio test for testing $\mu_1 = \mu_2$ vs $\mu_1 \neq \mu_2$.
   (b) Show that this is equivalent to the two sample $t$-test.

4. (10) Consider a machine with three components. The failure times of the components are iid exponential($\lambda$). The machine will continue to work as long as at least two of the components work. Find the expected time of failure of the machine.

5. (10+5+5+5) Let $X_1, \cdots, X_n$ be iid Poisson($\lambda$) random variables. We assume a Gamma($\alpha, \beta$) prior on $\lambda$.
   (a) Derive the posterior distribution of $\lambda$ and find the posterior mean $\hat{\lambda}$.
   (b) Show that $\hat{\lambda}$ is a consistent estimator for $\lambda$.
   (c) Show that $\sqrt{n}(\hat{\lambda} - \lambda)$ converges in distribution to $\mathcal{N}(0, \lambda)$.
   (d) Find an asymptotic $(1 - \alpha)$ level confidence interval for $\lambda$ using (c).